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JOURNAL OF PURE AND APPLIED SCIENCES

Travelling Wave Solutions for the Space and Time β −Fractional Chen-Lee-Liu Equation Using Tanh- Coth Method

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Abstract: In this study, we derive certain travelling wave solutions for the space and time β −fractional Chen-Lee-Liu equation, which serves as a fundamental equation in optical fiber modeling. It has many applications in wide variety of fields such as in the study of nonlinear dynamics, circuit design, signal processing, encryption and decryption of chaotic signals to mention a few. The tanh-coth scheme has been implemented to the space and time β −fractional Chen-Lee-Liu model equation to achieve the exact travelling wave solutions. The study also presents the necessary constraint conditions for the existence of soliton solutions. The obtained wave profiles might play important role in fiber optics, nonlinear optics and telecommunications systems. Furthermore, numerical simulations are illustrated for some of the obtained results, through 3D and 2D graphs.

Keywords: Chen-Lee-Liu equation, Travelling wave solutions, Fractional Chen-Lee-Liu equation, space and time β −fractional derivative.

1. INTRODUCTION

The investigation of optical soliton solutions within the realm of fiber optic pulse propagation remains a vibrant area of research. Numerous models capturing this dynamic arise from diverse contexts. For instance, the Schrödinger-Hirota equation is examined for dispersive solitons, while the Fokas-Lennel equation is explored in scenarios characterized by low group velocity dispersion (GVD), among others. The wave phenomena of Chen-Lee-Liu equation (CLL) can be used in optical fiber. The signal pulse of the optical soliton solution (OSS) of the Chen-Lee-Liu equation can be discussed in the optical fiber. Clearly, most of these systems are typically described in the time domain and are described by the field propagation at different frequencies. Most dynamical systems have complex partial differential equations and focus on these equations in fiber optic communication systems. In addition, significant advances were made during this period, such as the development of fiber amplifiers nonlinear effects on optical fibers and optical solitons for transmitting data through optical fiber losses. Many scholars have studied the CLL equation and investigated the OSSs. In that sense, (Zhang *et al.*, 2015) studied CLL equation through the Darboux transformation that included higher order components and obtained rogue wave solutions. Yildirim (2019) reported the dark, bright, and singular solitons of the CLL equation using the trial equation scheme. Biswas *et al.* (2018) have explored chirped OSSs from the CLL equation by using the extended trial equation scheme. A complex envelope travelling wave method was applied to CLL equation and explored by (Triki *et al.*, 2018). Bansal *et al.* (2020) reported the dark, bright, type OSSs in the CLL equation using the lie symmetry analysis. Recently Rehman *et al.* (2021) and Akinyemi *et al.* (2021) investigated the new and explicit OSSs of the CLL equation by utilizing the new extended direct algebraic and generalized $\left(\frac{G'}{G}\right)$ -

expansion methods, Rayhanul Islam *et al.* (2023) investigated the optical soliton solutions, bifurcation, and stability analysis of the Chen-Lee-Liu model and so on.

In this paper, we delve into the fractional Chen-Lee-Liu equation utilizing the tanh-coth method to derive its precise traveling wave solutions. The remaining parts of the paper are as follows: the definition and some properties of beta fractional derivative has been explained in section 2. The tanh-coth technique has been discussed in section 3. In section 4, mathematical analysis of space and time beta fractional Chen-Lee-Liu (CLL) equation. Section 5 provides the application of tanh-coth method to solve fractional CLL equation. Section 6 explains results and discussion and finally conclude in section 7.

2. THE BETA DERIVATIVE

The concept of incorporating memory effects into mathematical modeling has been a longstanding challenge. Traditional models often lack a natural framework to accommodate memory, as highlighted in works by (Podlubny 1998; Oldham 1974), and Singh *et al.* (2017). Fractional derivatives, as introduced by Caputo *et al.* (1971, 2015) and (Atangana 2016; Atangana *et al.*, 2016), offer a comprehensive explanation for this memory effect. (Khalil *et al.*, 2014), introduced the "conformable derivative," which adheres to classical derivative properties such as the composite (chain rule), product rule, and quotient rule. Further analysis of this derivative was conducted by (Atangana *et al.*, 2013), who established related theorems. For additional insights into fractional derivatives, refer to works by (He *et al.*, 2017), (Abdeljawad *et al.*, 2015), (Chung, 2015) and (Yusuf, 2019). Atangana (2016) also introduced the "beta-derivative," which addresses several limitations of fractional derivatives and finds applications in modeling various physical problems.

The beta-derivative, as defined by Atangana (2016), is as follows:

$${}_0^A D_t^\beta f(t) = \lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-f(t)}{\varepsilon}, \tag{1}$$

Beta derivative possesses the following properties

1. $${}_0^A D_t^\beta (af(t)+bg(t)) = a\,{}_0^A D_t^\beta f(t) + b\,{}_0^A D_t^\beta g(t), \tag{2}$$
2. $${}_0^A D_t^\beta (k) = 0, \quad \text{for any constant } k. \tag{3}$$
3. $${}_0^A D_t^\beta \big(f(t).g(t)\big) = g(t)\,{}_0^A D_t^\beta f(t) + f(t)\,{}_0^A D_t^\beta g(t), \tag{4}$$
4. $${}_0^A D_t^\beta\left(\frac{f(t)}{g(t)}\right) = \frac{g(t)\,{}_0^A D_t^\beta f(t) - f(t)\,{}_0^A D_t^\beta g(t)}{g^2(t)}, \tag{5}$$

Consider $\varepsilon = \left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1} h, h \to 0$, when $\varepsilon \to 0$,

We have

5. $${}_0^A D_t^\beta f(t) = \left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\frac{df(t)}{dt}, \tag{6}$$

With

$\xi = wx - \frac{l}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta$, for time $\beta -$ fractional derivative, (7)

Where l and w are constants, and

6. $$\xi = \frac{w}{\beta}\left(x+\frac{1}{\Gamma(\beta)}\right)^\beta - \frac{l}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta, \tag{8}$$

for space and time $\beta -$ fractional derivative.

3 METHODOLOGY

3.1 Tanh-coth method

The partial differential equation (PDE) given by

$$P(u, u_t, u_x, u_{xx}, u_{xxx}, \ldots) = 0, \tag{9}$$

can be converted into an ordinary differential equation (ODE) $Q(u', u'', u''', \ldots) = 0$. (10)

Using the wave variable transform $\xi = x - ct$.

Equation (10) is then integrated as long as all terms contain derivatives, with the integration constants taken to be zero.

Introducing the new independent variable

$$Y = \tanh(\mu\,\xi)\xi = x - ct, \tag{11}$$

Where μ is the wave number, leads to the change of

derivatives:
$\frac{d}{d\xi} = \mu(1 - Y^2)\frac{d}{dY}$,
(12)
$\frac{d^2}{d\xi^2} = -2\mu^2 Y(1 - Y^2)\frac{d}{dY} + \mu^2(1 - Y^2)^2\frac{d^2}{dY^2}$.
(13)
And so on.
The tanh-coth method allows for the finite expansion:
$(\mu\,\xi) = S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k}$,
(14)
Where M is a positive integer, typically determined through the balancing method. We usually balance the highest derivative and the highest order of the nonlinear term in the equation.
By substituting Eq. (14) into the reduced ODE, we collect all coefficients of each power of Y^k, $0 \le k \le nM$ in the resulting equation. These coefficients must vanish, resulting in a system of algebraic equations involving the parameters $a_k, b_k, \mu, and\ c$.
Finally, through this process, we obtain an analytic solution $u(x,t)$ in closed form.

4. MATHEMATICAL ANALYSIS

In this section we explore the mathematical analysis of space and time $\beta -$ fractional CLL model equation:

4.1. Space and time β −fractional Chen-Lee-Liu Equation

We examine the progression of a slowly varying envelope represented by a family of Chen-Lee-Liu equations (CLL), as formulated in Atangana (2013) and further explored by (Yusuf *et al.*, 2019).
$i\,{}_0^A D_t^{\beta} u + a\,{}_0^A D_x^{2\beta} u + ib(|u|^2)\,{}_0^A D_t^{\beta} u = 0.$
(15)
In the above equation, $u(x,t)$ represents the normalized electric-field envelope, while ${}_0^A D_t^{\beta}$ and ${}_0^A D_x^{\beta}$ denote beta derivatives as defined by Atangana (2016). The coefficients a and $b$correspond to the group velocity dispersion and the Bohm potential, respectively, which are significant in studying chiral solitons with quantum Hall effect.
To solve Eq. (15), we start with transformation
$u(x,t) = u(\zeta)e^{i\Theta(x,t)}$,
(16)
$u(x,t)$, represents the shape of the pulse so that
$\zeta = \frac{l}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} - \frac{\vartheta}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}$,
(17)
And the phase component is given by
$\Theta(x,t) = -\frac{k}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \varphi_0(\zeta)$.
(18)
Let k represent the soliton frequency and ω signify the wave number of the soliton. The function $\varphi_0(\zeta)$is an additional phase function that depends on the variable ζ and ϑ denotes the soliton's speed. By substituting Eq. (16) into Eq. (15) and separating the real and imaginary components, we derive the following results:
$-\omega u + \vartheta u\Theta' + au'' - au\Theta'^2 - ak^2 u + 2aku\Theta' - bu^3\Theta' + bku^3 = 0,$ (19)
And
$a(u\Theta'' + 2u'\Theta') - \vartheta u' - 2aku' + bu^2 u' = 0,$
(20)
Where $u' = \frac{du}{d\zeta}, u'' = \frac{d^2u}{d\zeta^2}, \Theta' = \frac{d\Theta}{d\zeta}, \Theta'' = \frac{d^2\Theta}{d\zeta^2}$.
To solve the equations above, we employ the following ODE of the form
$\Theta' = z_1 u^2 + z_2$.
(21)

Where z_1 and z_2 are the nonlinear and constant chirp parameters to be determined. By substituting Eq. (21) into Eq. (20), we derive two algebraic equations that determine these chirp parameters.
$z_1 = -\frac{b}{4a}, z_2 = k + \frac{\vartheta}{2a}$.
(22)
Substituting Eq. (20) along with Eq. (21) into Eq. (19) yields the following result:
$u'' + A_1 u + A_2 u^3 + Au^5 = 0,$
(23)
Where
$A_1 = \frac{\vartheta^2}{4a^2} + \frac{\vartheta k}{a} - \frac{\omega}{a}, A_2 = -\frac{b\vartheta}{2a^2}, A_3 = \frac{3b^2}{16a^2}$.
(24)
Applying the balancing principle in Eq. (23) results in $M = \frac{1}{2}$, which is not in a closed form. To achieve closed-form solutions, we employ the transformation:
$u = V^{\frac{1}{2}},$
(25)
Putting the above Eq. (25) in Eq. (23), we obtain
$4A_1V^2 + 4A_2V^3 + 4A_3V^4 + 2VV'' - V'^2 = 0.$

(26)
Applying the balancing principle in Eq. (26) gives $M = 1$.

5 APPLICATIONS

This section deals with the Tanh-coth method to get the precise travelling wave solutions of the Chen-Lee-Liu equation.

5.1 The solution of space and time β −fractional Chen-Lee-Liu equation

In this part, we utilize the tanh-coth method to derive the solution of the equation referred to as Eq. (15).
We aim to find the solution of Eq. (26) in the following form:

$$V(\xi) = a_0 + a_1 Y(\xi) + b_1 Y^{-1}(\xi).$$
(27)

By substituting $V(\xi)$ and its derivatives in eq. (26) and setting the coefficients of $Y^j: j = -1,0,1$ to zero, it produces a system of algebraic equations:

$$12A1a_1^2b_1 + 8A_1a_0a_1 + 48A_3a_0a_1^2b_1 + 12A_2a_0^2a_1 - 4\mu^2a_0a_1 + 16A_3a_0^3a_1 = 0,$$
$$24A_3a_0^2a_1^2 + 12A_2a_0a_1^2 + 16A_3a_1^3b_1 + 6\mu^2a_1b_1 - 2\mu^2a_1^2 + 4A_1a_1^2 = 0,$$
$$4A_1a_1^3 + 4\mu^2a_0a_1 + 16A_3a_0a_1^3 = 0,$$
$$3\mu^2a_1^2 + 4A_3a_1^4 = 0,$$
$$48A_3a_0a_1b_1^2 + 12A_2a_0^2b_1 + 16A_3a_0^3b_1 + 12A_2a_1b_1^2 - 4\mu^2a_0b_1 + 8A_1a_0b_1 = 0,$$
(28)
$$-2\mu^2b_1^2 + 16A_3a_1b_1^3 + 6\mu^2a_1b_1 + 4A_1b_1^2 + 24A_3a_0^2b_1^2 + 12A_2a_0b_1^2 = 0,$$
$$16A_3a_0b_1^3 + 4\mu^2a_0b_1 + 4A_2b_1^3 = 0,$$
$$4A_3b_1^4 + 3\mu^2b_1^2 = 0,$$
$$48A_3a_0^2a_1b_1 + 24A_2a_0a_1b_1 + 4A_1a_0^2 + 4A_2a_0^3 + 4A_3a_0^4 - \mu^2a_1^2 - \mu^2b_1^2 + 24A_3a_1^2b_1^2 - 12\mu^2a_1b_1 + 8A_1a_1b_1 = 0.$$

Solving this system Eq. (28), using maple software package.

We obtain the exact solutions as

$$a_0 = -\frac{3}{8}.\frac{A_2}{A_3}, a_1 = 0, b_1 = \frac{3}{8}.\frac{A_2}{A_3}, \mu = \pm\frac{1}{4}.\sqrt{-\frac{3}{A_3}}.A_2,\ A_1 = \frac{3}{16}.\frac{A_2^2}{A_3}$$
(29)

$$u_{1,1}(x,t) = -\frac{3}{8}.\frac{A_2}{A_3} + \frac{3}{8}.\frac{A_2}{A_3}coth\left(\frac{1}{4}.\sqrt{-\frac{3}{A_3}}.A_2\left[\frac{l}{\beta}\left(x+\frac{1}{\Gamma(\beta)}\right)^\beta - \frac{\nu}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta\right]\right) \quad (30)$$

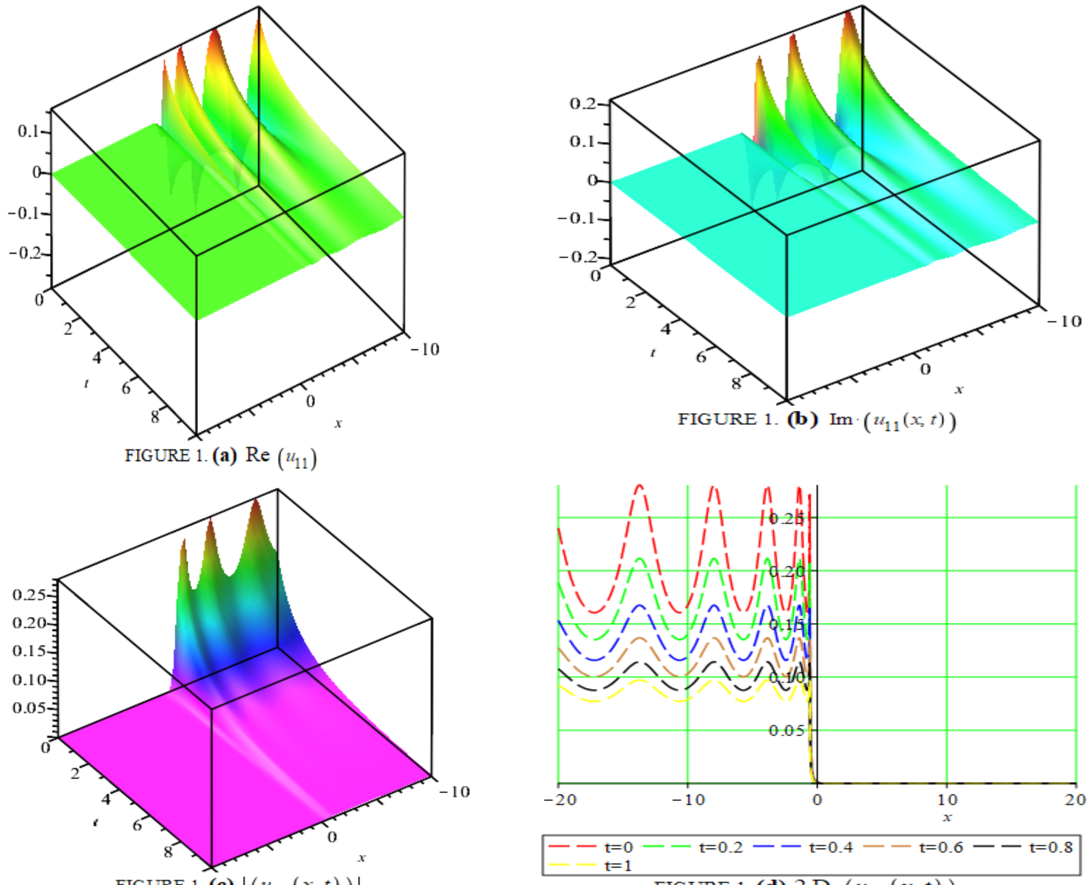

FIGURE 1. (a) Re (u_{11})

FIGURE 1. (b) Im $\cdot(u_{11}(x,t))$

FIGURE 1. (c) $|(u_{11}(x,t))|$

FIGURE 1. (d) 2 D$\cdot(u_{11}(x,t))$

FIGURE 1. (a) 3D-plot of the real, (b) 3D-plot of the imaginary, (c) 3D-plot of modulus parts of the exact travelling wave solution of $u_{11}(x,t)$. (d) 2D-plot of the exact travelling solution of $u_{11}(x,t)$. For the values $A_1 = 1, A_3 = -1, A_1 = \frac{3A_2^2}{16A_3}, l = 4, v = -1, \beta = 0.5$.

$$u_{2,1}(x,t) = -\frac{3}{8}.\frac{A_2}{A_3} + \frac{3}{8}.\frac{A_2}{A_3} tanh\left(\frac{1}{4}.\sqrt{-\frac{3}{A_3}}.A_2\left[\frac{l}{\beta}\left(x+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \frac{v}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right]\right). \quad (31)$$

$$A_2 = 1, A_3 = 1, A_1 = \frac{3}{16}.\frac{A_2^2}{A_3}, l = 2, v = -1, \beta = \frac{1}{2} \quad (32)$$

$$u_{3,1}(x,t) = -\frac{3}{8}.\frac{A_2}{A_3} + \frac{3}{8}.\frac{A_2}{A_3} tanh\left(\frac{1}{4}.\sqrt{-\frac{3}{A_3}}.A_2\left[\frac{l}{\beta}\left(x+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \frac{v}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right]\right). \quad (33)$$

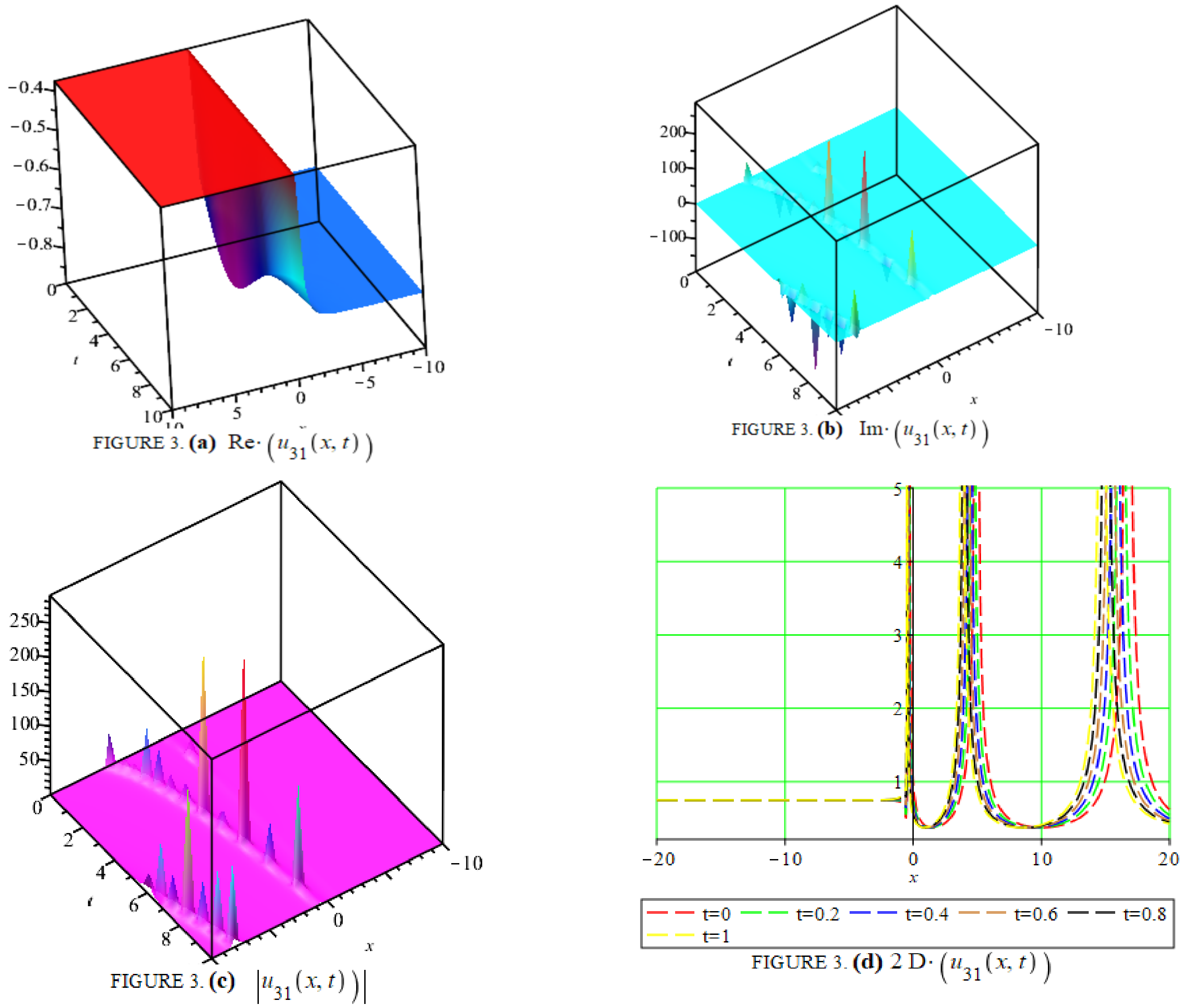

FIGURE 3. **(a)** Re$\cdot(u_{31}(x,t))$

FIGURE 3. **(b)** Im$\cdot(u_{31}(x,t))$

FIGURE 3. **(c)** $|u_{31}(x,t))|$

FIGURE 3. **(d)** 2 D$\cdot(u_{31}(x,t))$

FIGURE 3. (a) 3D-plot of the real, (b) 3D-plot of the imaginary, (c) 3D-plot of modulus parts of the exact travelling wave solution of $u_{31}(x,t)$. (d) 2D-plot of the exact travelling solution of $u_{31}(x,t)$, at $t = 0, t = 0.2, t = 0.4, t = 0.6, t = 0.8, t = 1$. For $A_1 = 1, A_2 = 1, A_2 = 1, A_1 = \frac{3A_2^2}{16A_3}, l = 2,\ v = -1,\ \beta = 0.5$.

$$a_0 = -\frac{3A_2}{8A_3}, a_1 = -\frac{3A_2}{8A_3}, b_1 = 0, A_1 = \frac{3A_2^2}{16A_3}. \quad (34)$$

$$u_{4,1}(x,t) = -\frac{3}{8}\cdot\frac{A_2}{A_3} + \frac{3}{8}\cdot\frac{A_2}{A_3 tanh\left(\frac{1}{4}\sqrt{-\frac{3}{A_3}}A_2\xi\right)}. \quad (35)$$

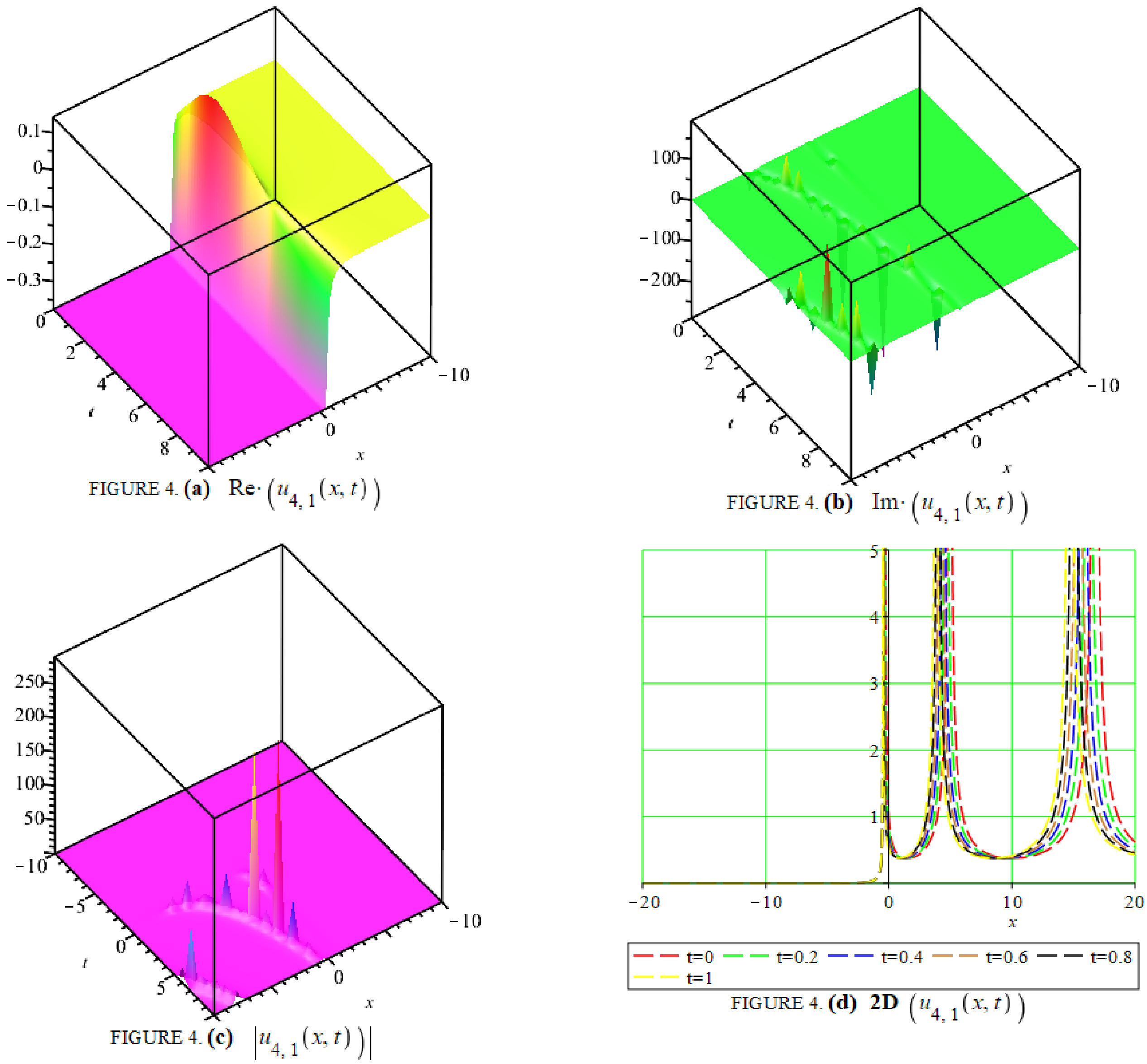

FIGURE 4. (a) 3D-plot of the real, (b) 3D-plot of the imaginary, (c) 3D-plot of modulus parts of the exact travelling wave solution of $u_{4,1}(x,t)$. (d) 2D-plot of the exact travelling solution of $u_{4,1}(x,t)$, at $t=0, t=0.2, t=0.4, t=0.6, t=0.8, t=1$. For $A_1=1, A_2=1, A_2=1, A_1=\frac{3A_2^2}{16A_3}, l=2,\ v=-1,\ \beta=0.5$.

$$u_{5,1}(x,t)=-\frac{3A_2}{8A_3}+\frac{3A_2}{16A_3}tanh\left(\sqrt{-\frac{3}{64A_3}}.A_2.\xi\right)+\frac{3A_2}{16A_3}coth\left(\sqrt{-\frac{3}{64A_3}}.A_2.\xi\right),$$

(36)

$$A_1=\frac{3A_2^2}{16A_3}.$$

(37)

$$u_{6,1}(x,t)=-\frac{3A_2}{8A_3}-\frac{3A_2}{16A_3}tanh\left(\sqrt{-\frac{3}{64A_3}}.A_2.\xi\right)-\frac{3A_2}{16A_3}coth\left(\sqrt{-\frac{3}{64A_3}}.A_2.\xi\right), \quad (38)$$

$$A_1=\frac{3A_2^2}{16A_3}.$$

(39)

$$u_{7,1}(x,t)=\frac{3A_2}{8A_3}-\frac{3IA_2}{16A_3}tanh\left(\frac{\sqrt{3}\sqrt{\frac{1}{A_3}}}{8}.A_2.\xi\right)+$$

$$\frac{3IA_2}{16A_3} coth\left(\frac{\sqrt{3}\sqrt{\frac{1}{A_3}}}{8}.A_2.\xi\right),$$

(40)

$$A_1 = \frac{15A_2^2}{64A_3}.$$

(41)

RESULTS AND DISCUSSION

The tanh-coth method is employed to establish exact travelling wave solutions and other solitons for the space and time beta fractional Chen-Lee-Liu equation. We obtained seven nontrivial solutions of which four numerical simulations were reported. Complex structures of solution $u_{1,1}(x,t)$ Eq. (30) in Figure 1. (a), multiple soliton solutions of $u_{2,1}(x,t)$ Eq. (31) in Figure 2. (c), kink structure depicted by $u_{3,1}(x,t)$ Eq. (33) in Figure 3. (a), anti- kink shape by the solution $u_{4,1}(x,t)$ Eq. (35) presented in Figure 4. (a). The solutions are in form of hyperbolic and complex functions. The plain understanding for the physical features and mechanisms to the reported solutions by suitable choice of parameter values are shown through 3D both real, imaginary, and modulus as well as in 2D plots.

CONCLUSION

In this work, we have investigated exact travelling wave solutions like hyperbolic and complex solutions to space and time fractional Chen-Lee-Liu equation with tanh-coth method. These solutions are favourable for understanding diverse nonlinear physical phenomena. The structure of the solutions was shown to be kink, anti-kink, multiple solitons and other complex shapes. The constraints conditions for the existence of soliton solutions are reported. The obtained results exhibited that the proposed approach is powerful, efficient and can be used to extract exact travelling wave solutions for other nonlinear partial differential equations that appear in various fields like engineering, optical fibers, oceanography, mathematical biology to mention a few.

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